LVIII. *A treatiſe on the preceſſion of the equinoxes, and in general on the motion of the nodes, and the alteration of the inclination of the orbit of a planet to the ecliptic. Inſcribed to the gentlemen of the* Royal Society, *by M.* De St. Jaques Silvabelle.

Tranſlated from the French *M. S. by* J. Bevis, *M. D.*

INTRODUCTION.

Read March 12, 1752.

IF the earth were perfectly ſpherical, the action of the ſun on all the parts which compoſe it, would not produce any effect to make it turn round its centre; becauſe the moment, which would be produced on one ſide, would be always counterbalanced by an equal moment on the oppoſite ſide of the centre.

It would be the ſame, if the earth were a ſpheroid flatted at the poles, and the ſun was always in the equator, or in the ninetieth degree of declination: But in every other degree of declination its action on the exceſs of matter about the equator has a tendency to make the equator approach towards the ſun's place, or to diminiſh the angle of the ſun's declination, by making the earth's axis to turn round its centre in the plane of the circle of the ſun's declination.

The earth has then, at every inſtant, two motions of rotation; one about the axe of the equator, called alſo the earth's axe; and this is the diurnal motion, which is uniform; the other motion of rotation is performed about the axe of the circle of the ſun's declination, which is a diameter of the equator; and this motion is produced by the action of the ſun on the redundant matter about the equator, and is continually accelerated, from the continual application of the ſolar action producing it.

The point E, *fig.* 1. *n*° 2. which is the interſection of the circumference of the equator and the circumference of the circle of declination, has two motions, whoſe directions are perpendicular to each other. Let Ee be the ſpace, which it runs through in an inſtant dt, in the circumference of the equator, by the uniform diurnal motion, and let $E\varepsilon$ be the ſpace it runs through in the ſame inſtant, in the circumference of the circle of declination, by an accelerated motion, as has been explained.

The point E, in vertue of theſe two motions Ee and $E\varepsilon$, will not circulate either in the circumference $EeQe'QE$ of the equator, or in the circumference $E\varepsilon PE'P'E$ of the circle of declination, but forming the rectangular parallelogram $Eee'\varepsilon$, the diagonal Ee' will be the elementary arc of the circumference $EeqE'$, in which the point E will circulate, and the angle eEe' will be equal to the angle QCq, and equal to the angle $P'Cp'$ which the pole P' runs through in an inſtant in the circumference $P'p'QqPQ'P'$ whoſe plane is perpendicular to the plane of the circle of declination; and when the lines Ee, $E\varepsilon$ are known at every inſtant, $P'p'$ will alſo be known, ſince the angle eEe' = to the angle QCq = to the angle $P'Cp'$.

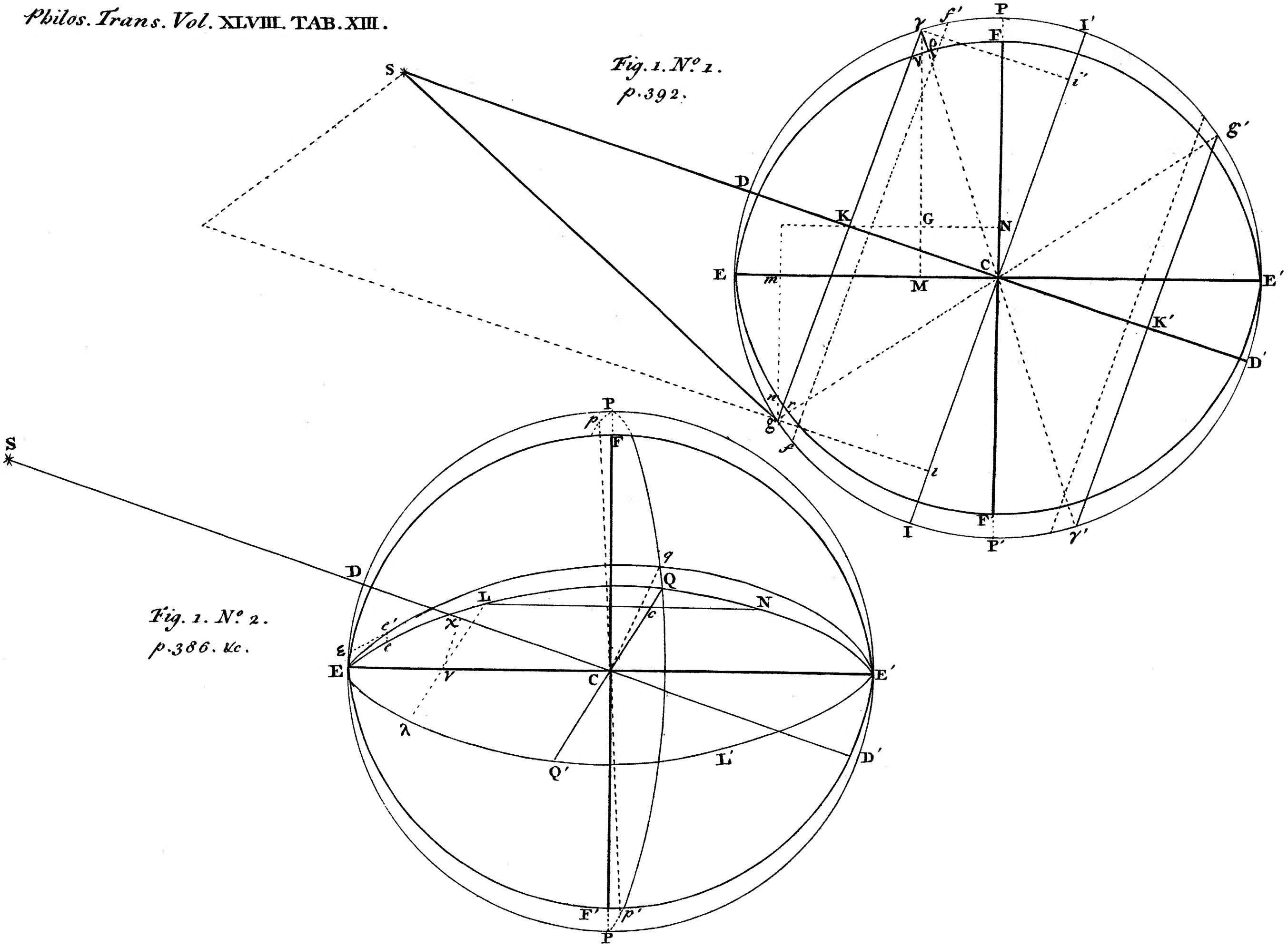

Fig. 1. N° 1.
p. 392.

Fig. 1. N° 2.
p. 386. &c.

The inſtantaneous motion of the pole, which is $P'p'$, or Pp, *fig.* 2. *n*° 1. & *n*° 2*. may be reſolved into two, PR and PM, perpendicular to each other, and both to the earth's axe. The former cauſes the pole P to move parallel to the ecliptic ♈ ♋ ♎ ♑, and alters the place of the ſolſtice ♋, and conſequently alſo that of the equinoctial points ♈ and ♎; the latter, which is according to PM, alters the inclination of the earth's axe to the ecliptic.

To have the motion of the pole parallel to the ecliptic, or, which is the ſame, the motion of the node ♈, or the preceſſion, in the ſame time that the ſun paſſes from the equinox ♈ to the ſolſtice ♋, take the integral of the lines PR, ſuppoſing PR generally to expreſs the inſtantaneous preceſſion for any declination of the ſun S.

And to have the alteration of the inclination in the ſame time that the ſun is paſſing from ♈ to ♋, take the ſum, or the integral of the lines PM, ſuppoſing PM generally to expreſs the inſtantaneous alteration of the inclination of the earth's axe to the ecliptic for any given declination of the ſun.

The ſum of the lines PR is always the ſame, and has the ſame ſign, or the ſame direction, during every quarter of the ſun's revolution, whether he moves from ♈ to ♋, or from ♋ to ♎, or from ♎ to ♑, or from ♑ to ♈; ſo that the preceſſion anſwering to any one quarter of the ſun's revolution about the earth, or to three months, being known, that multiplied by 4 will be the annual preceſſion; by 8 will give it for two years; by 16 for 4 years, &c.

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* *Fig.* 2. is explained at the beginning of *Prob.* 5.

Likewiſe the ſum of the lines *P M* is ever the ſame for every quarter of the ſolar revolution; but it has alternatively a contrary ſign; that is, a contrary direction. During the quarter from ♈ to ♋, the alteration of the inclination of the earth's axe to the ecliptic is poſitive, and the angle of the inclination increaſes; but during the ſucceeding quarter, or from ♋ to ♎, the alteration of the inclination is negative, and the angle of the inclination diminiſhes: And as the diminution from ♋ to ♎ is equal to the augmentation from ♈ to ♋, it follows, that at the end of the ſemi-revolution the inclination of the earth's axe to the plane of the ecliptic will become again the ſame, having undergone an oſcillation, which is completed in a ſemirevolution. It is the ſame, when the ſun paſſes from ♎ to ♈. The angle of the inclination increaſes from ♎ to ♑, and decreaſes from ♑ to ♈, where it becomes again the ſame it was at ♎.

And hence the inclination of the earth's axe to the ecliptic may be conſidered as conſtant, tho' ſubject to this oſcillation, and indeed to ſeveral others, which will be preſently explained, they being all regular, and performed in regular periods.

The earth's inclination to the ecliptic being conſtant, and the motion of the pole which produces the preceſſion, being always parallel to the plane of the ecliptic, the earth's pole moves in a parallel to the ecliptic, about 23 degrees and a half diſtant from the pole of the ecliptic, and the terreſtrial axe deſcribes a conic ſurface.

To this motion of the terreſtrial axe or pole is to be aſcribed the apparent motion of the ſtars about the pole of the ecliptic.

But hitherto we have not conſidered, that to the preceſſion, thus cauſed by the ſun, we are to add that likewiſe produced by the moon; and it remains, that we examine into the motion of the earth's pole, cauſed by the action of the moon on the redundant matter about the earth's equator.

All, that has been ſaid concerning the ſun, is alike applicable to the moon, which we may put in the place of the ſun; the moon's orbit in the place of the ecliptic; and the time of the moon's revolution round the earth in the place of the revolution of the ſun round the earth: And we ſhall find the motion of the earth's pole parallel to the lunar orbit, which is always the ſame at every quarter of the time of the revolution of the moon round the earth, and the oſcillation of the earth's axe to the plane of the lunar orbit, which is completed, in each ſemirevolution of the moon round the earth.

But whereas the plane of the lunar orbit, which is always inclined to the plane of the ecliptic in an angle of about 5 degrees, never continues in a conſtant poſition, like the plane of the ecliptic, ſo that its pole deſcribes a ſmall circle parallel to the ecliptic, at the diſtance of about 5 degrees from the pole thereof; it follows, that the preceſſion, with reſpect to the lunar orbit, is not the ſame as with reſpect to the ecliptic; and that the motion of the pole parallel to the lunar orbit ſhould be referred to the plane of the ecliptic: Which is done by reſolving the motion of the pole, parallel to the plane of the lunar orbit, into two motions, the one parallel to the plane of the ecliptic, and the other perpendicular thereto, and in the plane of the ſolſticial colure.

The former of theſe two motions gives the preceſſion with reſpect to the ecliptic, and has its direction always the ſame way.

The latter motion has two oppoſite directions, in the two ſemirevolutions of the pole of the lunar orbit round the pole of the ecliptic, and cauſes an oſcillation of the terreſtrial axe on the plane of the ecliptic, which is compleated in a revolution of the pole of the lunar orbit round the pole of the ecliptic.

From all that has been ſaid, it follows, that there are five diſtinct motions of the pole of the earth; namely, two of preceſſion, which are parallel to the plane of the ecliptic, and three of oſcillation on the plane of the ecliptic.

The two of preceſſion are cauſed, the one by the ſun, the other by the moon. That, which is cauſed by the ſun, is conſtantly the ſame at every quarter of the time of the revolution of the ſun round the earth, that is, every three months: That which is cauſed by the moon, is conſtantly the ſame at every quarter of the time of the revolution of the moon round the earth; that is, about every ſeven days.

Of the three motions of oſcillation, one is cauſed by the ſun, and is completed in the time of the ſemirevolution of the ſun round the earth, taken from one equinox to the following one; that is, in ſix months.

The other is cauſed by the moon; and each oſcillation is completed in the ſpace of a ſemirevolution of the moon round the earth; that is, in about 14 days.

The third is cauſed likewiſe by the moon, and ariſes from the plane of her orbit being different from the plane of the ecliptic, and from the pole of the lunar orbit making its revolution about the pole of the ecliptic in about 18 years and two thirds. And this oſcillation is compleated in the time of the revolution of the pole of the lunar orbit about the pole of the ecliptic; that is, in about 18 years and two thirds.

It will appear in the memoir, that there is a relation purely geometrical between the quantity of the nutation, during the time of the ſemirevolution of the pole of the lunar orbit, and the quantity of the preceſſion, cauſed likewiſe by the moon in the ſame time. This relation is quite independent of the force of the moon, of the quantity of the earth's flatneſs, of the quantity of the terreſtrial matter, and, in a word, of every thing of a phyſical nature that can enter into the problem.

We are content to examine the motions of the pole of the earth produced by the ſun and the moon. The ſame method, and the ſame *formulæ*, will give likewiſe the motions of the terreſtrial pole ariſing from any other planet, as Saturn, Jupiter, *&c.* but theſe motions are too minute to merit attention.

Whatever has been ſaid of the action of the ſun on the redundant matter about the earth's equator, is alſo applicable to his action on a ſimple ring placed at the equator, without adhering to the terreſtrial globe; and the motion of the pole of ſuch ring may be determined by the ſame method, and conſequently the motion of its nodes on the plane of the ecliptic, and the alteration of the inclination of its axe to the ſame plane. And ſince theſe motions are the ſame, whether the ring be ſuppoſed entire, or a ſmall portion of it only be conſidered, or a mere point thereof, the motions of the nodes, and the alteration of the inclination of a moon, or a ſatellite of a planet, may thereby be known. And the *formulæ* differ in nothing from thoſe of the motion of the nodes of the earth's equator, and of the alteration of the obliquity of the earth's axe to the plane of the ecliptic, but in this; that the action of the ſun on the ring to make it turn, is exerted entirely thereon; whereas in the problem of the preceſſion this force muſt neceſſarily be diſtributed throughout the whole maſs of the earth, on account of the adherence of the ring to the globe of the earth.

DIVISION of the WORK.

This memoir is divided into four ſections.

The 1ſt ſection treats of the motion of the pole of the terreſtrial equator cauſed by the ſun.

The 2d ſection treats of the motion of the pole of the terreſtrial equator cauſed by the moon.

The 3d ſection treats of the motion of the pole of a ring, or of the orbit of a moon, cauſed by the ſun.

The 4th ſection contains the application of the *formulæ* found in the other ſections.

SECT. I.

Of the motion of the pole of the terreſtrial equator cauſed by the action of the ſun.

PROBLEM I.

Article 1. *To find the moment, which reſults from the attraction of each particle of the earth towards the ſun, in the inverſe ratio of the ſquare of the diſtance, to make the earth's axis turn upon its centre* C.

Let $E F E' F E$, *fig.* 1. *n°* 1. be the ſection of the earth by the plane of the circle of the ſun's declination;

tion; $P E P' E' P$ the circumference whoſe diameter is the earth's equator, $E E'$ or $P P'$. Let $E E'$ be perpendicular to $P P'$, and in the equator. Let $I I'$ be perpendicular to $C S$, and aſſume this diameter $I I'$ for the leaver to which all the moments are to be referred.

The motion of any point, as g, towards the ſun, is $\frac{S}{\overline{Sg}^2}$. By reſolving this motion into two, one according to $g C$, the other parallel to $C S$; the motion according to $g C$ has no tendency to make the point g turn round the centre C. The motion of the point g parallel to $C S$ will be $\frac{S}{\overline{Sg}^2} \times \frac{SC}{Sg}$ or $\frac{S \times SC}{\overline{Sg}^3}$; but the motion of the centre C according to CS is $\frac{S}{\overline{SC}^2}$; therefore the relative motion of the point g, in regard gard to the centre C, is $\frac{S \times SC}{\overline{Sg}^3} - \frac{S}{SC}$ or $S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{SC}^3}\right)$.

The moment of the point g to turn about the centre C, will therefore be $S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{SC}^3}\right) \times g \times CI$, the moment being the product of the motion by the maſs of the body, and by the arm of the leaver. And this moment cauſes the point I to approach towards the ſun, when it is poſitive, or when $S C$ is greater than $S g$; and cauſes the point I to recede from the ſun when it is negative, or when $S g$ is greater than $S C$.

It will be found, in like manner, that the moment of the point g' to turn about the centre C, is $S \times SC \times \left(\frac{1}{\overline{Sg'}^3} - \frac{1}{\overline{SC}^3}\right) \times g' \times CI'$; or with regard to the point I which is on the oppoſite ſide of the centre to the point I', this moment, by changing the ſigns, will be, $S \times SC \times \left(\frac{1}{\overline{SC}^3} - \frac{1}{\overline{Sg'}^3}\right) \times g' \times CI' = S \times SC \times \left(\frac{1}{\overline{SC}^3} - \frac{1}{\overline{Sg'}^3}\right) \times g \times CI$. Therefore the moment of the two equal points g and g' to cauſe the point I to turn about the centre C, is $S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{SC}^3}\right) \times g \times CI + S \times SC \times \left(\frac{1}{\overline{SC}^3} - \frac{1}{\overline{Sg'}^3}\right) \times g \times CI =$ $g \times CI \times S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{Sg'}^3}\right)$.

If the points γ and γ' be taken equally diſtant from the points D and D', as the points g and g', it will likewiſe be found, that the moment of the equal points γ and γ' to cauſe the point I to turn about the centre C, is $\gamma \times \overline{CI} \times S \times \overline{SC} \times \left(\frac{1}{\overline{S\gamma'}^3} - \frac{1}{\overline{S\gamma}^3}\right) = \gamma \times$ $\overline{CI} \times S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{Sg'}^3}\right)$.

Therefore the moment of all the four points together, g, g', γ, γ', to cauſe the point I to turn about the centre C, is $(g - \gamma) \times \overline{CI} \times S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{Sg'}^3}\right)$, which becomes $= 0$ when the point g can be taken $= \gamma$, as may be done throughout the whole extent of the circle $PEP'E'P$; but if the zone $EP'E'PEF'E'FE$ be taken away from this circle, to have the true figure of the earth, we ſhall have $g = \overline{gf} \times \overline{gr}$, and $\gamma = -\overline{\gamma f} \times \overline{\gamma \varrho}$, taking the points g and γ for the elements of the zone, and $g - \gamma = \overline{gf} \times \overline{\overline{\gamma\varrho} - \overline{gr}}$, becauſe $\overline{\gamma f} = \overline{gf}$.

Let the lines CS and CP, or CE, be called s and a, reſpectively; the radius 1, or unity; the ſine of the angle SCP, V; its coſine, or the ſine of the angle SCE, u; CK, x; gK, y; $\overline{gm}$, z; γM, z'; $\overline{gf}$ or $\overline{\gamma f'}$, du; PF, $\mathrm{a}a$; we ſhall have $gn = \mathrm{a}z$; $\gamma v = \mathrm{a}z'$ (by the property of the ellipſe); alſo calling gr, r; and $\gamma\varrho$, ϱ; and regarding the little triangle grn as a right-lined one, and ſimilar to the triangle gCm, we ſhall have $r = \frac{z}{a} \times \mathrm{a}z$, and likewiſe $\varrho = \frac{z'}{a} \times \mathrm{a}z'$.

By the proportion of ſines to the ſides of triangles, we ſhall have $\gamma G = yV$: $CN = xu$. Finally, if we conſider the point S, or the ſun, as at an infinite diſtance, we ſhall have $Sg = SK = SC - CK = s - x$, and $Sg' = SK' = SC + CK' = s + x$. Therefore $\frac{1}{\overline{Sg}^3} = \frac{1}{s^3 - 3s^2x + 3sxx - x^3}$, and $\frac{1}{\overline{Sg'}^3} = \frac{1}{s^3 + 3s^2x + 3sx^2 + x^3}$. Therefore $\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{Sg'}^3} =$

$\frac{6s^2x + 2x^3}{(s - x)^3 \times (s + x)^3}$; and, rejecting the infinitely ſmall magnitudes, $= \frac{6x}{s^4}$.

Therefore the moment wherewith the four points g, g', γ, γ', cauſe the point I to turn, will, by ſubſtituting in the place of the letters their analytical values, be $(\varrho\, du - r\, du) \times S \times y \times \frac{6x}{s^3}$; or putting, inſtead of ϱ and r, the values above found, we ſhall have for the moment $\left(\frac{\mathrm{a}}{a} z'z' - \frac{\mathrm{a}}{a} zz\right) \times du \times S \times y \times \frac{6x}{s^3} = \frac{6\,\mathrm{a}\,S}{s^3 a} \times y\, x\, du \times (z' + z) \times (z' - z)$. But $z' + z = \gamma M + gm = 2\,\overline{\gamma G} = 2\,yV$; and $z' - z = \gamma M - gm = 2\,\overline{GM} = 2\,\overline{CN} = 2\,xu$. Therefore $(z' + z) \times (z' - z) = 4\,Vu\,xy$; and the moment is $\frac{6\mathrm{a}S}{s^3} \times 4\,Vu \times x\,xy^2\,du = \frac{6\mathrm{a}S}{s^3} \times 4\,Vu \times xx\,dx\,u$. But in the circle DgI we have $y\,du = a\,dx$, and the moment is $\frac{6\mathrm{a}S}{s^3} \times 4\,Vu \times xxy\,dx = \frac{6\mathrm{a}S}{s^3} \times 4\,Vu \times dx \sqrt{aa - xx}$. But the integral of $x^2\,dx \sqrt{aaxx}$ is $\frac{1}{4} aa \int dx \sqrt{aa - xx} - \frac{x}{4}(aa - xx)^{\frac{3}{2}}$, and when $x = a$, this integral is $\frac{1}{4} aa \int dx \sqrt{aa - xx}$, or, calling the ratio of the circumference to the diameter π, we ſhall have $\int dx \sqrt{aa - xx}$, or the area of the quadrant whereof a is the radius, $= \frac{1}{2} a\pi \times \frac{1}{2} a = \frac{1}{4} a^2 \pi$. Therefore the integral of $\frac{6a'S}{s^3} \times 4\,Vu \times xx\,dx$

$\sqrt{aa - xx}$, or the ſum of the moments of all the points which compoſe the elliptic zone $E\,P'\,E'\,P\,E\,F\,E\,F\,E$ is $\frac{6aS}{s^3} \times \frac{Vu\pi a^4}{4}$.

If the terreſtrial ſpheroid be cut by any plane parallel to the plane $P\,E\,P'\,E'\,P$ of the circle of declination, *fig.* 1. *n*° 2. the ſection will be an ellipſe ſimilar to the ellipſe $E\,F'\,E'\,F\,E$; and if the greater ſemiaxe $c'\,L$ of this ellipſe be called X, it will be found, as has been already in the ellipſe $E\,F'\,F'\,F\,E$, that the ſum of the moments of all the points of this ellipſe to turn about its centre C, is $\frac{6aS}{S^3} \times \frac{Vu\pi X^4}{4}$.

It has not been taken into conſideration, that here the centre S of the ſun is a little below the plane of the ſection, the line $c'\,S$ making an inſenſible angle with the line $C\,S$.

Calling $C\,c'$, Y; the moment of all the ſections parallel to $E\,F'\,E'\,F\,E$ to turn about the axe $C\,c'$ will be the integral of $\frac{3aS}{S^3} \times \frac{Vu\pi X^4}{2} \times dY$, or the integral of $\frac{3aS}{s^3} \times \frac{Vu\pi}{2} \times (aa\,YY)^2 \times dY = \frac{3aS}{s^3} \times \frac{Vu\pi}{2} \times (a^4\,dY - 2a^2\,Y^2\,dY + Y^4\,dY)$, and this integral is $\frac{3aS}{s^3} \times \frac{Vu\pi}{2} \times (a^4\,Y - \frac{2}{3}\,a^2\,Y^3 + \frac{1}{5}\,Y^5)$; and when $Y = a$, the integral is $\frac{3aS}{s^3} \times \frac{Vu\pi}{2} \times \frac{8}{15}\,a^5$. This is the ſum of the moments of all the points of the hemiſpheroid of the earth formed by the ſection $E\,P'\,E'\,P\,E$ of the circle of declination, and the ſum of the moments of all the points of the whole ſpheroid to turn about the axe QQ' is $\frac{3aS}{s^3} \times Vu\pi \times \frac{8}{15}a^5$. Which was to be found.

COROLLARY.

2. If it were required to find the ſum of the moments of all the points of a ſingle crown $ELQE'Q'E$, *fig.* 1. *n*° 2. placed at the equator, and detached from the earth, to turn about the axe QQ' of the circle of the ſun's declination, the thickneſs of the crown being a a:

The moment of the point L to turn about c' is the ſame as that of the point γ to turn about C; as is manifeſt from the reſolution of the motion according to SL into two, $L\gamma$, and γS*; and from the great diſtance of the ſun γS will not differ from LS; the triangle $L\gamma S$ being right-angled at γ.

By the preceding problem, if $c'L$, or CY, be called X; and Cc', Y; and all the other denominations of that problem be retained; we ſhall have $\gamma\varkappa = Xu$, and $C\varkappa = XV$.

And the moment of the point γ is $\gamma \times \gamma\varkappa \times S \times \overline{SC} \times \left(\frac{1}{\overline{S\gamma}^3} - \frac{1}{\overline{SC}^3}\right) = \gamma \times Xu \times S \times \frac{3XV}{s^3}$.

And putting inſtead of γ, the element of the crown, which is a $a \times du$, calling, in this caſe, du the element of the circumference $ELQE'$, the moment of an element of the crown will be a$a \times \frac{3S}{s^3} \times Vu \times$

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* The lines SL and γS are not drawn in the *fig.* to avoid confuſion.

$XXdu$. But in the circle $EQE'E$ we have, $Xdu = a\,dY$; therefore the moment of an element of the crown is $\mathrm{a}a \times \frac{3S}{s^3} \times Vu \times a \times dY = \frac{3S}{s^3} \times \mathrm{a}a^2\, Vu \times dY\sqrt{aa - YY}$; and the ſum of the moments of the quadrant ELQ of the crown is $\frac{3S}{s^3} \times \mathrm{a}a^2\, Vu \times \frac{1}{2} a\pi \times \frac{1}{2} a = \frac{3S}{s^3} \times \mathrm{a}\, Vu \times \frac{1}{4}\pi a^4$. And every other quadrant of the crown having an equal motion, the ſum of the moments of all the points of the crown, to turn, in vertue of the ſolar action, about the axe QQ', is $\frac{3S}{s^3} \times \mathrm{a}\, Vu \times \pi a^4$.

PROBLEM II.

3. *To find the ſum of the moments of all the points of the terreſtrial ſpheroid turning about the axe* QQ', *the motion of the point* E *being given, and* $= \mu$.

We muſt firſt ſeek the ſum of the moments of all the points of the ellipſe $EF'E'FE$, *fig.* 1. *n*° 2. turning about its centre C; and to have the ſum of the moments of all the points of this ellipſe, we ſhall firſt find the ſum of the moments of all the points of the circle $EP'E'PE$, and then we ſhall ſubduct therefrom the ſum of the moments of the whole elliptic zone $EP'E'PEF'E'FE$.

The motion of any point of the circle $EP'E'PEC$, placed at the diſtance x from the centre C, will be $\frac{x}{a}\mu$; the motions being here in the ſame ratio as the diſtances from the centre, ſince all the points complete their revolution in the ſame time; and the moment of all the points which compoſe the circumference $2\,\pi\,x$ will be $2\,\pi\,x \times \frac{x}{a}\,\mu \times x$; the moment being produced from the maſs by the motion, and by the arm of the leaver.

Therefore the motion of all the concentric circumferences which compoſe the circle, is $\int \frac{2\,\pi\,\mu}{a} \times x^3\,d\,x = \frac{1}{2}\,\frac{\pi\,\mu}{a} \times x^4 = \frac{1}{2}\,\frac{\pi\,\mu}{a} \times a^4$, when $x = a$, as for the circle $E\,P'\,E'\,P\,E$.

To have the ſum of the moments of all the points of the elliptical zone $E\,P'\,E'\,P\,E\,F'\,E'\,F\,E$, all the points of the zone may be conſidered as placed at the diſtance a, from the centre C, and having, conſequently, the ſame motion as the point E, becauſe the greateſt thickneſs $P\,F$ of the zone is very ſmall.

The ſum of the moments of the zone will be equal to the quantity of matter multiplied by the motion μ, and by the arm of the leaver a, or $2\,a\,\pi \times \frac{1}{2}\,\mathrm{a}\,u \times \mu \times a = \pi\,\mu\,\mathrm{a}\,a^3$.

Therefore the ſum of the moments of all the points of the ellipſe $E\,F'\,E'\,F\,E$ turning about its centre C, the motion of the point E being μ, will be $\frac{1}{2}\,\frac{\pi\,\mu}{a} \times a^4 - \pi\,\mu\,\mathrm{a}\,a^3$.

It will likewiſe be found generally, that the ſum of the moments of a ſimilar ellipſe, of which $C'\,L$, or X, is half the greater axe, and which turns about its centre C', is $\frac{1}{2}\,\frac{\pi\,\mu}{a} \times X^4 - \frac{\pi\,\mu\,\mathrm{a}}{a} \times X^4$.

Therefore the ſum of all the ſimilar ellipſes which compoſe the terreſtrial hemiſpheroid, is $\int \frac{1-2\mathrm{a}}{2a} \times \pi\,\mu \times X^4\,dY = \frac{1-2\mathrm{a}}{2a} \times \pi\,\mu \times \int (aa - YY)^2 \times dY$, which, as has been ſeen at the end of *Prob.* I. when, after the integration Y has been made $= a$, will be $\frac{1-2a}{2a}\,\pi\,\mu \times \frac{8}{15}\,a^5$, and multiplying by 2, the ſum of all the points of the terreſtrial ſpheroid will be $(1-2\mathrm{a}) \times \pi\,\mu \times \frac{8}{15}\,a^4$. Which was to be found.

COROLLARY.

4. If you would find the ſum of the moments of all the points of the ring $EQE'Q'B$, *fig.* 1. *n*° 2. turning about the axe QQ':

The moment of an element L of the crown placed at the diſtance X from the axe of rotation QQ', is $\mathrm{a}\,a\,du \times \frac{\mu X}{a} \times X$, which is the product of the quantity of matter by the motion, and by the arm of the leaver. But in the circle $EQE'E$ we have, $du\frac{a\,dY}{X}$; therefore the moment of an element of the crown is $\mathrm{a}\,a\,\mu \times X\,dY$, whoſe integral, which is $\mathrm{a}\,a\,\mu \times a\,a\,\pi$, gives the ſum of the moments of all the points of the crown. Which was to be found.

PROBLEM III.

5. *Having given, the moment wherewith the earth turns about the axe* QQ' *of the circle of the ſun's declination, To find the motion of the pole* P, *or, which is the ſame, of the point* E *in the plane of the circle of the ſun's declination.*

By *Prob.* I. the moment wherewith the earth turns about the axe CQ, by the action of the ſun, is $\frac{3S}{s^3} \times \mathrm{a}\,\pi \nu u \times \frac{8}{15} a^5$.

By *Prob.* II. the moment wherewith the earth turns about the ſame axe CQ, the velocity of the point P, or of the point E, being μ, is $(1-2\mathrm{a}) \times \pi \mu \times \frac{8}{15} a^4$.

We ſhall have then $\frac{3S}{s^3} \mathrm{a}\,\pi \nu u \times \frac{8}{15} a^5 = (1-2\mathrm{a}) \times \pi \mu \times \frac{8}{15} a^4$; whence we get $\mu = \frac{3S\mathrm{a}a\nu u}{s^3 \times (1-2\mathrm{a})}$. And this is the motion of the pole P, or of the point E, about the centre C. Which was to be found.

COROLLARY.

6. In like manner, the moment wherewith the crown $EQE'Q'E$ turns about the axe QQ' being given, the motion of the point E may be known. For,

By the *Coroll.* of *Prob.* I. the moment of this crown, turning about the axe CQ by the action of the ſun, is $\frac{3S}{s^3} \times \mathrm{a}\,\pi \nu u \times a^4$.

By the *Coroll.* of *Prob.* II. the moment of this crown is alſo $\mathrm{a}\, a \mu \times a^2 \pi$.

We have then $\frac{3S}{s^3} \times \mathrm{a}\, \pi \nu u \times a^4 = \mathrm{a}\, a \mu \times a^2 \pi$; whence we get $\mu = \frac{3S}{s^3} \times a \nu u$. Which was to be found.

Remark. It is proper to obſerve, that the motion of the point *E* of the crown is the ſame, whether the crown be entire, or there be no more of it but the point *E*.

For, by *Prob.* I. the motion of any point *g*, parallel to *C S*, *fig.* 1. *n*° 2. and reſpectively to the centre *C*, is $S \times \overline{SC} \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{SC}^3}\right)$, which is $= S \times \frac{3x}{S^3}$; and when the point *g* becomes the point *E*, we have $x = a\nu$, ν being the coſine of the angle *D C E*. Therefore the reſpective motion of the point *E* parallel to the motion *C S* of the centre *C*, is $S \times \frac{3a\nu}{S^3}$; and reſolving this motion into two, the one according to *E C*, and the other perpendicular to *E C*, this latter will be $= S \times \frac{3a\nu}{s^3} \times \frac{au}{a} = \frac{3S}{s^3} a \nu u$, being the ſame as that which was found for the point *E* when it is united to the crown.

This ſhews that the motion of the other points of the crown produce no alteration in the motion of the point *E*, which is owing to this; that all the points of the crown have the ſame angular motion about the axe *C Q*.

PRO-

PROBLEM IV.

7. *To find at every inſtant the variation of the place of the pole of the earth.*

The earth has two motions of rotation, the one about the axe of the equator, which is likewiſe the earth's axis; and the other about the axe CQ of the circle of the ſun's declination; this motion is cauſed by the ſun's action on the redundant matter about the equator.

The point E, *fig.* 1. *n*° 2. which is the interſection of the circumference $EQE'Q'E$ of the equator and the circle $PEP'E'P$ of the ſun's declination, will therefore have at every inſtant two motions, whoſe directions are perpendicular to each other.

The former of theſe two motions, which is the diurnal motion, is uniform; and if we call it m, the ſpace run through in an inſtant, dt, is $m\,dt$; the motion being always equal to the ſpace divided by the time, or the ſpace equal to the product of the time. Alſo let Ee be that ſpace.

The ſecond motion of the point E which is performed in the circumference $EPE'P'E$ of the circle of the ſun's declination, and ariſes from the action of the ſun, is continually accelerated, from the continual application of the ſun's action; and if we call the initial motion μ, at the firſt term of the inſtant, dt, or the increment of the motion, the motion at the end of the inſtant dt, is $\mu\,dt$; and the ſpace gone thro' uniformly by the motion during the inſtant dt, is $\mu\,dt \times dt$, or $\mu\,dt^2$. Let $E\varepsilon$ be that ſpace.

The point E, in conſequence of theſe two motions together, $E e$ and $E \varepsilon$, circulates neither in the circumference $E \mathcal{Q} E' \mathcal{Q}' E$ of the equator, nor in the circumference $E P E' P' E$ of the ſun's declination. But if we form the rectangle $E e e' \varepsilon E$, the diagonal $E e'$ will be the elementary arc of the circumference $E e' q E' E$ wherein the point E will circulate; and the angle $e E e'$, equal to the angle $\mathcal{Q} c q$, equal to the angle $P' C p'$, will be the angle whereby the pole P' is elevated, and the pole P depreſſed, below the circle of declination, by moving in the circumference $P' p' \mathcal{Q} q P p \mathcal{P}'$, whoſe plane is perpendicular to the plane of delination; and the point p' is the true place of the pole at the end of the inſtant $d t$, and the ſmall arc $P' p'$ expreſſes the inſtantaneous variation of the place of the earth's pole. Which was to be found.

COROLLARY I.

8. The ſimilar ſectors $e E e'$, $\mathcal{Q} C q$, or $P' C p'$, whoſe three ſides of the one are each parallel to the three ſides of the other, give this proportion, $E e$: $e e'$ or $E \varepsilon$ :: $C P'$: $P' p' = \frac{E \varepsilon}{E e} \times C P'$ = (by the preceding *Prob.*) to $\frac{\mu d t^2}{m d t} \times C P' = \frac{\mu d t}{m} \times a$. But by *Problem* III. $\mu = \frac{3 S \mathrm{a} a \nu u}{s^3 (1 - 2\mathrm{a})}$; therefore $P' p' =$ $\frac{3 S \mathrm{a} a \nu u}{s^3 \times m \times (1 - 2\mathrm{a})} \times a d t = A \nu u a d t$, making A $= \frac{3 S \mathrm{a} a}{s^3 \times m \times (1 - 2\mathrm{a})}$.

COROLLARY II.

9. If, inſtead of the terreſtrial ſpheroid, we ſuppoſed only a ſingle ring, it would be found in like manner, that the pole P of ſuch ring would ſhift its place during the inſtant $d\,t$, by running through the arc $\mathcal{P}'\,p' = \frac{E\,\varepsilon}{E\,e} \times CP' = \frac{\mu\,d\,t}{m} \times a$. But then we ſhould take the value of μ as in *Coroll.* of *Prob.* III. which is $\frac{3S}{s^3} \times a\,\nu\,u$. Therefore we ſhall have $P'\,p' = \frac{\mu\,d\,t}{m} \times a = \frac{3\,S\,a}{s^3 \times m} \times \nu\,u\,a\,d\,t$.

To apply, then, the *formula* of the preced. *Coroll.* it will ſuffice to make $\frac{a}{1 - 2a}$ in that *Coroll.* $= 1$. for then the *formula* of *Coroll.* I. becomes the ſame with that juſt now found in this *Coroll.*

This will be of uſe hereafter for applying the *formulæ* of the motion of the nodes of the terreſtrial equator to the motion of the nodes of a moon or ſatellite of a planet.

COROLLARY III.

10. It ſhould be well obſerved in the preceding *Coroll.* that the direction $P'\,p'$ of the pole P' is ever parallel to the direction $E\,e$ of the point E of the equator, ſince they are both perpendicular to the plane of the circle of the ſun's declination, and the direction $P'\,p'$ goes on the ſame ſide as the direction $E\,e$ of the point E, which is placed on the ſame ſide of the

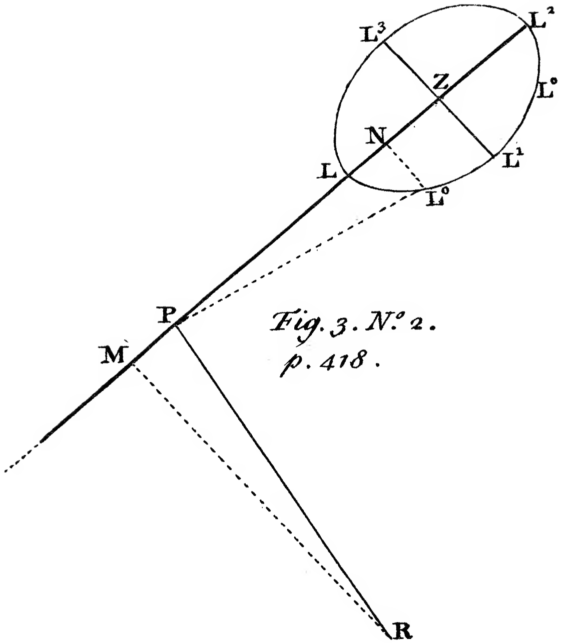
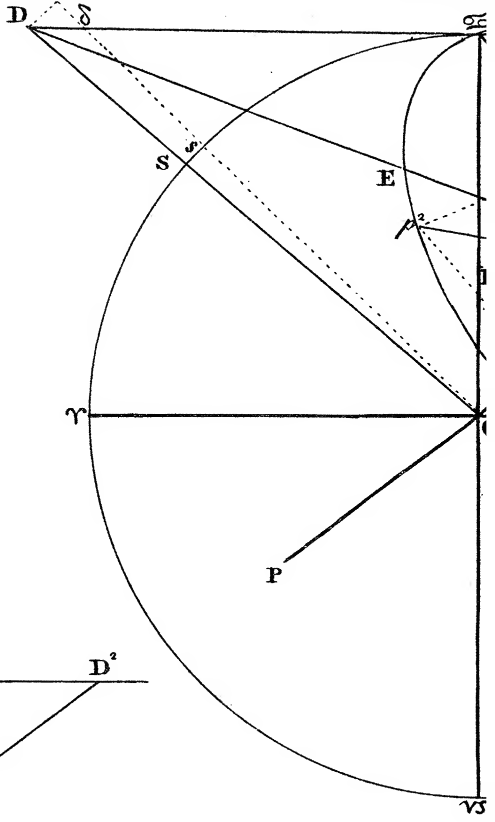


Fig. 3. N.° 2.
p. 418.



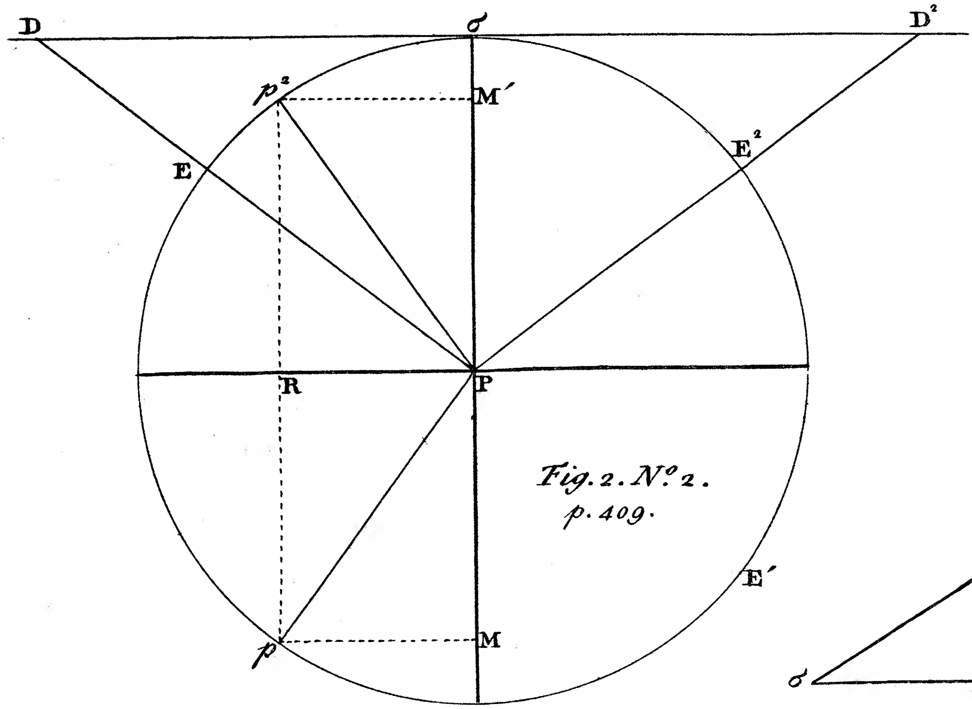


Fig. 2. N.° 2.
p. 409.

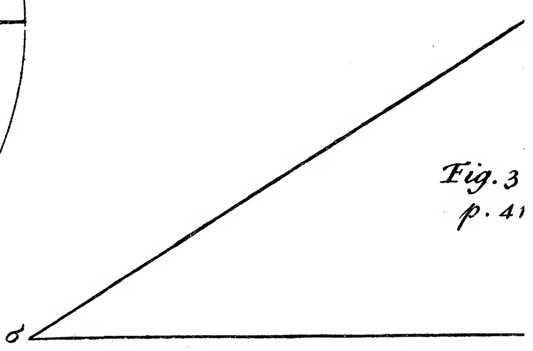


Fig. 3
p. 41

Philos. Trans. Vol. XLVIII. TAB. XIV.

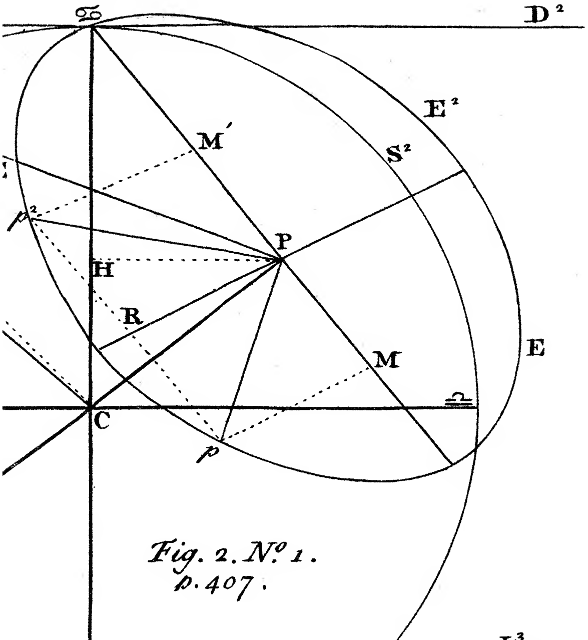


Fig. 2. N.° 1.
p. 407.

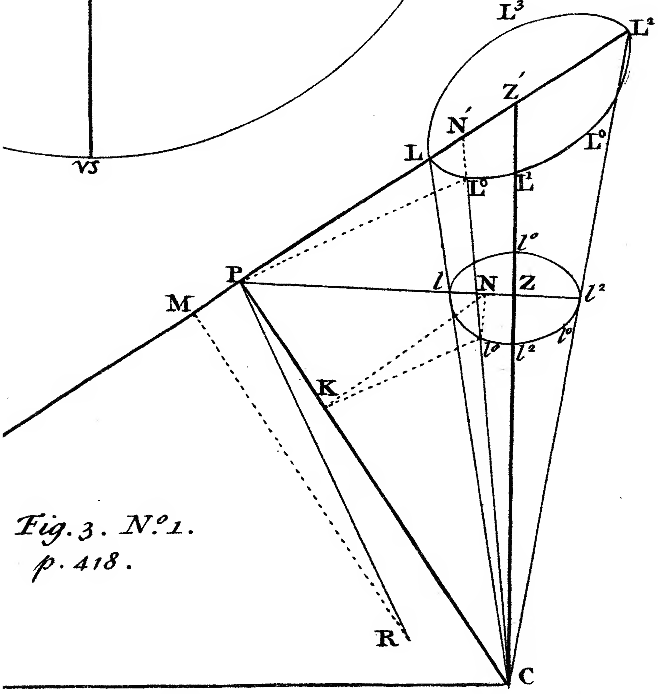


Fig. 3. N.° 1.
p. 418.

J. Mynde ſculp.

the pole *P′*, with reſpect to the line *S C D′*, and the direction of the oppoſite pole *P*, goes on the ſame ſide as the direction of the point *E* of the equator which is placed on the ſame ſide of the pole *P* with reſpect to the line *S C D′*. This ſhould be carefully attended to in regard to the following *Problem.*

PROBLEM V.

11. *To determine at every inſtant the motion of the pole of the earth with reſpect to the ecliptic.*

P P′, *fig.* 2. *n*° 1. *&* *n*° 2*. is the earth's axe; ♈ ♋ ♎ ♑ repreſents the ecliptic; ♋ *E p E* ♋ is the tangent plane to the pole *P*, and parallel to the equator; the line *D* ♋ D^2 is the interſection of this plane with that of the ecliptic, and is tangential at the common point ♋ to the circles ♋ *E′ p E* ♋ and ♈ ♋ ♎ ♑. The angle *P C* ♋ is the angle of inclination of the earth's axe to the plane of the ecliptic, being likewiſe the ſolſticial angle. The earth, by its diurnal motion, turns about the axe *P′ P* according to the order of ♋ *E′ p E* ♋; and the ſun, by the annual motion in the ecliptic, moves according to the order of ♈ ♋ ♎ ♑.

Let *S* be any place of the ſun in the ecliptic. By *Art.* 10. the motion of the pole *P* making the acute angle *P C S*, will be according to *P p* perpendicular to the plane *P C S D P*, and conſequently to the line *P D*.

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[*] *Note*, That the *fig.* *n*° 2. is only to repreſent the circle ♋ *E p* ♋, which could not be repreſented but by projection in *fig.* *n*° 1.

Forming, therefore, the rectangle *P M p R*, the motion of the pole according to *P p* will be reſolved into two, according to *P R* and *P M*.

The motion of the pole according to *P R* is parallel to the plane of the ecliptic, and to the line of the nodes ♈ *C* ♎. This motion tends to move the pole *P* about the axe of the ecliptic, and in a parallel to the ecliptic, and cauſes the equinoctial point ♈ to recede, and the point ♑, and the node ♎, to advance towards the ſun; which produces a preceſſion of the equinox equal to the angular motion of the pole *P* about the axe of the ecliptic; for it is indifferent whether the pole *P* of the earth moves about the axe of the ecliptic, whilſt the plane ♈ ♋ ♎ ♑ remains fix'd; or the ſame be ſuppoſed to be fix'd, and the plane of the ecliptic ♈ ♋ ♎ ♑ to turn about its centre *C*, and the axe of the ecliptic.

If we take another place S^2 of the ſun at the ſame diſtance from the ſolſtice ♋, but on the ſide contrary to *S*, the direction of the pole *P* will be, by *Art.* 10. according to Pp^2, perpendicular to the plane PCS^2D^2P, and conſequently to the line PD^2; and the ſun's declination being the ſame in D^2 as in *D*, the motion Pp^2 will be equal to the motion *P p*.

Reſolving the motion according to Pp^2 into two, according to *P K* and *P M'*, the motion according to *P R* produces the ſame preceſſion, and with the ſame ſign, or the ſame direction, as when the ſun was on the other ſide of the ſolſtice ♋, and in D; and the preceſſion is ever the ſame at every quarter revolution of the ſun.

The motion according to *P M'* cauſes the ſame variation in the inclination of the earth's axe to the ecliptic, as when the ſun was on the other ſide in S, but with a contrary ſign, or direction; and the ſum of the lines PM', from the ſolſtice ♋ to ♎, will be the ſame as the ſum of the lines PM, from ♈ to ♋, but with a contrary ſign: ſo that the angle of inclination of the earth's axe to the plane of the ecliptic is continually increaſing from ♈ to ♋, and then decreaſing from ♋ to ♎; and as the diminution is equal to the increaſe, the inclination becomes again the ſame in ♎ as it was in ♈, having undergone an oſcillation whoſe period is ſix months.

Whilſt the ſun paſſes from ♎ to ♑, there are the ſame motions of the pole, as whilſt it was paſſing from ♋ to ♎.

Thus is the motion of the pole determined at every inſtant, with reſpect to the ecliptic. Which was to be done.

PROBLEM VI.

12. *To find the quantity of the preceſſion, or, which is the ſame, the arc of the parallel to the ecliptic run through by the pole of the earth* P *in the ſpace of time the ſun is paſſing from the equinox to the ſolſtice, or from the ſolſtice to the equinox; which may be taken for a quarter of the time of the ſun's revolution about the earth; the angle comprehended between the equinox and the enſuing ſolſtice differing but by an infinitely ſmall quantity from 90 degrees.*

Calling, as in *Prob.* I. the ſemi-axe CP, a; *fig.* 2. *n*° 1. *&* *n*° 2. the ſine of the angle PCS, or PCD, V; and its coſine, or the ſine of the ſun's declination, u:

Let $P\text{♋}$, be b; $C\text{♋}, r$; we ſhall have $CD = \frac{a}{u}$; $PD = \frac{a}{u}V$; the difference Ss of the arc ♈ S, which is the ſun's diſtance from the node ♈, will be $\frac{rr \times d\overline{CD}}{\overline{CD} \times \overline{\text{♋}D}}$, as is found by theſe two proportions, $r : \overline{Ss} :: CD : D\omega = \frac{CD \times \overline{Ss}}{r}$, and $D\omega = \frac{CD \times \overline{Ss}}{r} : \delta\omega = d\overline{CD} : \overline{C\text{♋}} = r : \overline{\text{♋}D}$; whence we have $\overline{CD} \times \overline{Ss} \times \overline{\text{♋}D} = rr \times d\overline{CD}$, and $\overline{Ss} = \frac{rr \times d\overline{CD}}{\overline{CD} \times \overline{\text{♋}D}} = \frac{r\,du}{\sqrt{\frac{aa}{uu} - rr}}$ (putting inſtead of $\overline{CD}$ and $\overline{SD}$, their values $\frac{a}{u}$ and $\sqrt{\frac{aa}{uu} - rr}$) $=$ to the ſpace run through by the ſun S in the ecliptic in an inſtant, $= 2\pi r \times \frac{dt}{T}$, calling T the time the ſun employs in running thro' the whole circumference $2\pi r$.

Whence we have $dt = \frac{T}{2\pi r} \times \frac{r\,du}{\sqrt{aa - \frac{uu}{rr}}} = \frac{T}{2\pi} \times \frac{du}{\sqrt{aa - \frac{uu}{rr}}}$.

By *Art.* 8. we have $aPp = AVu \times adt$; therefore, by *Prob.* V. the inſtantaneous preceſſion will be $AVu \times adt \times \frac{\overline{PR}}{\overline{Pp}}$, or $AVu \times adt \times \frac{\overline{P\text{♋}}}{\overline{PD}}$, the triangles Ppr, $PD\text{♋}$, being ſimilar, ſince the three ſides of the one are perpendicular to the three ſides of the other. Putting, therefore, inſtead of $\overline{PD}$, $\overline{P\text{♋}}$, and dt, their values above found, we ſhall have the inſtantaneous preceſſion $= AVu \times \frac{ab}{\frac{aV}{u}} \times \frac{T}{2\pi} \sqrt{\frac{aa}{rr} - uu}$ $= Ab \times \frac{T}{2\pi} \times \frac{uu\,du}{\sqrt{\frac{aa}{rr} - uu}}$. But $\int \frac{uu\,du}{\sqrt{\frac{aa}{rr} - uu}} = -u\sqrt{\frac{aa}{rr} - uu} + \int du\sqrt{\frac{aa}{rr} - uu}$, or ſimply $= \int \frac{uu\,du}{\sqrt{\frac{aa}{rr} - uu}}$ when $u = \frac{a}{r}$, as it happens when the ſun is in the ſolſtice ♋, and then $\int du\sqrt{\frac{aa}{rr} - uu}$, which is the area of a quadrant of a circle whoſe radius is $\frac{a}{r}$, will be $\frac{a\pi}{2r} \times \frac{a}{2r} = \frac{aa\pi}{4rr}$.

Therefore the preceſſion, whilſt the ſun is paſſing from ♈ to ♋, is $Ab \times \frac{T}{2\pi} \times \frac{aa\pi}{4rr} = Ab \times \frac{T}{2} \times \frac{aa}{4rr}$;

and calling CH, which is = the radius of the circle parallel to the ecliptic which the pole P runs thro', g; we ſhall have $g = \frac{aa}{r}$, and the preceſſion $Ab \times \frac{\mathcal{T}}{2} \times \frac{aa}{4rr}$ will be likewiſe $Ab \times \frac{\mathcal{T}}{2} \times \frac{g}{4r}$. Which was to be found.

And this alſo is the quantity of the preceſſion during the time the ſun paſſes from the ſolſtice to the equinox, or generally, during the time of each quadrant of the ſun's revolution; as was ſeen in *Prob.* V.

Remark.

13. The difference of the ſun's diſtance S ♈ from the equinoctial point, is equal to the arc the ſun goes through in the ecliptic, when the node ♈ is fix'd, as was ſuppos'd in the foregoing *Prob.* but when the node ♈ moves in the ecliptic the contrary way to the ſun's motion, the difference of the ſun's diſtance from the node is equal to the arc gone through by the ſun, increaſed by the arc gone through by the node ♈.

Therefore always taking Ss, *fig.* 2. *nº* 1. for the difference of the ſun's diſtance from the node, we ſhall have $\overline{Ss}$ equal to the arc deſcribed by the ſun, more by the arc deſcribed by the node.

But, by the former *Prob.* we have ſeen, that the arc run through by the pole P in an inſtant dt, and in a parallel to the ecliptic, whereof g is the radius, is

$$AVu \times adt \times \frac{\overline{P\text{♋}}}{\overline{PD}} = AVu \times \frac{ab}{\frac{aV}{u}} \times dt = Abuudt.$$

The arc run through by the node ♈ will be then

$Abuudt \times \frac{r}{S}$, ſince theſe arcs, which meaſure equal angles, are as their radii g and r.

We ſhall have, therefore, in the above *Problem*, $\overline{Ss}$, or $\frac{r\,du}{\sqrt{\frac{aa-uu}{rr}}}$, $= 2\pi r \times \frac{dt}{T} + Abuu \times \frac{r}{g}\,dt$;

whence we get $dt = \frac{T\,du}{2\pi\sqrt{\frac{aa-uu}{rr}}} \times \frac{1}{1 + Abuu \times \frac{T}{2\pi g}}$.

But $\frac{1}{1+Abuu \times \frac{T}{2\pi g}} = 1 - Abuu \times \frac{T}{2\pi g} +$, &c. where the other terms may be neglected which contain the powers of A, becauſe A is a very ſmall fraction.

We ſhall have, therefore, $dt = \frac{T\,du}{2\pi\sqrt{\frac{aa-uu}{rr}}} \times \left(1 - Abuu \times \frac{T}{2\pi g}\right)$; and the inſtantaneous preceſſion will be $Ab \times \frac{T}{2\pi} \times \frac{uu\,du}{\sqrt{\frac{aa-uu}{rr}}} \times \left(1 - Abuu \times \frac{T}{2\pi g}\right)$ whoſe integral for the firſt term, in the ſolſticial point, where $u = \frac{a}{r}$, is $Ab \times \frac{T}{2\pi} \times \frac{aa\pi}{4rr}$, as was ſeen in the preceding *Problem*. But the integral of

$u^4\,du$

$$\frac{u^4\,du}{\sqrt{\frac{aa}{rr}-uu}} = -u^3\sqrt{\frac{aa}{rr}-uu} - \tfrac{3}{4}u \times \left(\frac{aa}{rr}-uu\right)^{\frac{3}{2}}$$

$+ 3\,\frac{aa}{rr}\int du\sqrt{\frac{aa}{rr}-uu}$, or ſimply, $= \frac{3}{4}\,\frac{aa}{rr}\int du$
$\sqrt{\frac{aa}{rr}-uu}$, when we ſuppoſe, after the integration,
$\frac{a}{r} = u$; and $= \frac{3}{4}\,\frac{aa}{rr} \times \frac{a\pi}{2r} = \frac{3}{4}\frac{aa}{rr} \times \frac{aa\pi}{4rr}$. Therefore the preceſſion from the ſun's departure from the equinox till his arrival at the ſolſtice, is $Ab \times \frac{T}{2} \times \frac{aa}{4rr} \times \left(1 - Ab \times \frac{T}{2\pi g} \times \frac{3}{4}\,\frac{aa}{rr}\right) = \left(g \text{ being } = \frac{aa}{r}\right) Ab \times \frac{T}{2r} \times \frac{g}{4} \times \left(1 - Ab \times \frac{T}{2\pi r} \times \frac{3}{4}\right)$.

When the motion of the node is not very ſenſible, it will ſuffice to make uſe of the *formula* of the preceding *Problem*, which is the firſt term of this: But when that motion is ſenſible, which is the caſe of the moon's node, when it is to be determined for a quarter of the revolution of the ſun, it will be requiſite to employ this *formula*, which contains two terms of the ſeries. Nay, it would be very eaſy to take a greater number of terms, ſo as to neglect nothing; but this would be abſolutely uſeleſs, becauſe of the quick converging of the ſeries.

PRO-

PROBLEM VII.

14. *To find the alteration of the inclination of the earth's axe to the ecliptic, during the ſun's paſſage from the equinox to the ſolſtice, or from the ſolſtice to the next equinox; which is the ſame, only affected with a contrary ſign, as has been ſeen in* Prob. V.

By *Art.* 8. we have $aPp = AVu \times adt$; and by *Prob.* V. the inſtantaneous variation of the inclination of the earth's axe to the ecliptic is $AVu \times adt \times \frac{PM}{Pp} = AVu \times adt \times \frac{\overline{\text{♋}D}}{\overline{PD}}$, becauſe of the ſimilar triangles PpM, $PD\text{♋}$, $=$ (taking the denominations of *Prob.* VI.) $AVu \times a \times \frac{T}{2\pi} \times \frac{r \times d\overline{CD}}{\overline{CD} \times \overline{\text{♋}D}} \times \frac{\overline{\text{♋}D}}{\overline{PD}} = AVu \times$

$$\frac{aT}{2\pi} \times \frac{rd\overline{CD}}{\overline{CD} \times \overline{PD}} = AVu \times \frac{aT}{2\pi} \times \frac{r \times \frac{du}{uu}}{\frac{ar}{u} \times \frac{a}{u}} = AV \times \frac{T}{2\pi}$$

$\times udu$, whoſe integral is $\frac{1}{2}Ar \times \frac{T}{2\pi} \times uu$; and when the ſun is at the point ♋, we have $1 : u :: r : a$, and $a = ru$; therefore the preceding integral is then $\frac{1}{2}A \times \frac{T}{2\pi r} \times aa$. Which was to be found.

Remark.

15. When the motion of the node is not inſenſible, you are to take the value of dt of the *Remark* that

follows *Art.* 12. which gives $dt = \frac{T}{2\pi} \times \frac{r \times d\overline{CD}}{\overline{CD} \times \overline{SD}} \times (1 - Ab \times \frac{T}{2\pi g} \times uu)$, and the inſtantaneous variation of the inclination of the earth's axe to the ecliptic will be $Ar \times \frac{T}{2\pi} \times udu \times (1 - Ab \times \frac{T}{2\pi g} \times uu)$, whoſe integral, when the ſun arrives at the ſolſtice, where we have $ru = a$, becomes $\frac{1}{2} Aaa \times \frac{T}{2\pi r} - \frac{1}{4} \frac{A \cdot a^4}{r^2} \times \frac{T}{2\pi r} \times Ab \times \frac{T}{2\pi g} =$ (becauſe $g = \frac{aa}{r}$) $\frac{1}{2} Aaa \times \frac{T}{2\pi r} \times (1 - \frac{1}{2} Ab \times \frac{T}{2\pi r})$; and this is the *formula* to be made uſe of when the motion of the nodes is ſenſible.

SECT II.

Of the motion of the pole of the equator of the earth cauſed by the action of the moon.

16. Whatever has been ſaid in *Sect.* I. of the preceſſion, and of the oſcillation of the earth's axe on the plane of the ecliptic, through the action of the ſun, is equally applicable to the action of the moon; it being requiſite only to ſubſtitute every-where the moon inſtead of the ſun; L inſtead of S, and λ inſtead of s; the time which the moon employs in completing her revolution about the earth, inſtead of the time the ſun employs in performing

his revolution about the earth, or *M* inſtead of *Tt* Laſtly, the lunar orbit, inſtead of the ecliptic. Likewiſe, inſtead of *A*, which (*Art.* 8.) was equal to $\frac{3S}{s^3 m}$, we muſt put $B = \frac{3L}{\lambda^3 m} \times \frac{\mathrm{a}a}{1-2\mathrm{a}}$.

17. We ſhall find then, by *Art.* 12. that the arc deſcribed by the pole *P* of the earth parallel to the plane of the lunar orbit, whilſt the moon makes one quarter of a revolution about the earth, is $Bb \times \frac{g}{2r} \times \frac{M}{4}$, which gives the arc deſcribed by the mean motion in an inſtant, dt, equal to $Bb \times \frac{g}{2r} \times dt = Bb \times \frac{aa}{2rr} \times dt$; putting, inſtead of g, its value $\frac{aa}{r}$.

18. We ſhall find alſo, by *Art.* 14. that the variation of the inclination of the earth's axe to the plane of the lunar orbit, when the moon is at its point of ſtation 90 degrees diſtant from the node of the equator with the lunar orbit, is $\frac{1}{2} B \times \frac{M}{2\pi r} \times aa$, whereby the angle of inclination of the earth's axe to the plane of the lunar orbit is increaſed when the moon is at its point of ſtation.

19. We therefore ſhould have nothing to add, if the plane of the lunar orbit were the ſame as that of the ecliptic, or if it always retained the ſame poſition: But as this plane continually varies its poſition, and its pole deſcribes a ſmall circle about the pole of the ecliptic, from whence it is always about 5 degrees diſtant; it ſtill remains that we examine what is the

preceſſion cauſed by the moon, with reſpect to the ecliptic; which may be done by reſolving the motion of the earth's pole, parallel to the plane of the lunar orbit, into two motions, one whereof is parallel to the plane of the ecliptic, and the other perpendicular thereto, and which varies the inclination of the earth's axe to the ecliptic. This is what we ſhall determine in the following *Problem.*

PROBLEM VIII.

20. *Having given, the motion of the pole* P *of the earth parallel to the plane of the lunar orbit, to find its motion parallel to the plane of the ecliptic.*

* *C P*, *fig.* 3. *n*° 1. & *n*° 2. is the ſemi-axe of the earth. The plane of the figure is the plane of the ſolſticial colure; *C Z* is the axe of the ecliptic, conſequently perpendicular to *C* ♋; ♋ the ſolſticial point, and *P C* ♋ the ſolſticial angle: *P Z* is parallel to *C* ♋, and perpendicular to *C Z*; and it is equal to the line *C H* of *fig.* 2. The circumferencc l, l', l^2, l^3, l is the circumference deſcribed by the pole, l, of the lunar orbit; L, L', L^2, L^3 is the ſection of the cone $C\,l$, $l'\,l^2\,l^3\,l$ prolonged, by the plane that is tangential at the pole of the earth, *P*.

Let l° be any place of the pole of the moon's orbit. If upon the point *P* be erected *P R*, perpendicular to the plane $P\,C\,l^{\circ}$, this perpendicular will be the direction of the motion of the earth's pole, parallel to the plane of the lunar orbit, the plane $P C l^{\circ}$ being perpendicular to that plane, $P C l^{\circ}$ being the axe thereof.

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* The lines which are in the plane which touches the earth's pole l, are ſeparately repreſented in *fig.* 2. *n*° 3.

Letting fall from the point R the perpendicular $R M$ upon the line ♋ $P Z$, the motion of the pole of the earth according to $P R$ may be reſolved into two; the one according to $P M$, the other parallel and equal to $R M$; and this latter expreſſes the preceſſion with regard to the ecliptic, and the former gives the alteration of the inclination of the earth's axe to the plane of the ecliptic.

The motion of the pole repreſented by $M R$, which gives the preceſſion with regard to the ecliptic, is always on the ſame ſide during the entire revolution of the pole of the lunar orbit, and is the ſame on either ſide the point l at the like diſtance from it; the motion according to $P R$ being then the ſame, and always on the ſame ſide, as has been explained in *Prob.* V.

* The motion according to $P M$ is the ſame on either ſide of the point l at the ſame diſtance from it, but with a contrary ſign, that is, in an oppoſite direction. For it is manifeſt, that if another point, as l°, be taken on the other ſide, and at an equal diſtance from the point l, the point M, where the perpendicular $R M$ falls, will be on the other ſide with regard to P, than the point M, ſince the angle $R P Z'$, which is here obtuſe, will then be acute, and leſs than the right angle of an angle equal to $L^{\circ} P Z'$.

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* The point l° placed on the other ſide of the point l, that is, between l and l^{3}, is not marked in the *fig.* nor the point M' where the perpendicular $K M'$ falls; for the ſake of not embarraſſing the *fg.* with too many lines, it being eaſy to conceive.

For this reaſon, the motion according to PM muſt produce an oſcillation of the earth's axis, whoſe period is the time of an entire revolution of the pole of the lunar orbit about the pole of the ecliptic.

PROBLEM IX.

21. *To find the quantity of the preceſſion cauſed by the moon with regard to the ecliptic; that is, the arc parallel to the ecliptic, run through by the earth's pole* P, *during the time of a ſemi-revolution* $l\,l^1\,l^2$ *of the pole* l *of the lunar orbit.*

Let there be drawn the lines NK, $l^\circ K$ perpendicular to the axe of the earth CP, and parallel to the lines NP, $l^\circ \mathcal{P}$, *fig.* 3. *n*° 1. & *n*° 2.

The triangles PRM, $\mathcal{P}NL^\circ$, KNl° are right-angled and ſimilar; for the three ſides of the firſt are perpendicular to the three ſides of the ſecond, and the three ſides of the ſecond are parallel to the three ſides of the third.

Still retaining the foregoing denominations, let $zl = x$; $ZN = z$; $\mathcal{P}Z = g$, as in *fig.* 2. the line CH; $CZ = f$; Cl or $Cl^\circ = C$.

We ſhall have $Nl^\circ = \sqrt{xx - zz}$; $NP = g - z$; $NK = \frac{f}{a} \times \overline{g - z}$, or $\frac{f}{a} \times \overline{g + z}$, when the point l° lies on the other ſide of l^1; $PR = \frac{g}{a} \times \overline{g - z}$, or $\frac{g}{a} \times \overline{g + z}$ when the point l° is on the other ſide of l^1. The triangle $Cl^\circ K$ is here ſimilar to the triangle whoſe three ſides are repreſented by the letters a, b, r, ſince $\frac{b}{r}$ expreſſes here the ſine of the angle of inclination of the earth's axe to the plane of the lunar orbit, and $\frac{a}{r}$ is its coſine; we ſhall have then $l^{\circ} K = \frac{ac}{r}$; $CK = \frac{bc}{r} = CP - PK = a - \frac{g}{a} \times \overline{g - z} = \frac{aa - gg + gz}{a} = \frac{ff + gz}{a}$, or $\frac{ff - gz}{a}$, when the point l° is on the other ſide with reſpect to l^{1}; $\frac{RM}{PR} = \frac{NK}{l^{\circ} K} = \frac{\frac{f}{a} \times \overline{g - z}}{\frac{ac}{r}}$, or $\frac{\frac{f}{a} \times \overline{g + z}}{\frac{ac}{r}}$ for the other ſide of l^{1}; $\frac{\overline{PM}}{PR} = \frac{N l^{\circ}}{l^{\circ} K} = \frac{\sqrt{xx - zz}}{\frac{ac}{r}}$: Laſtly, the difference of the arc $l^{1} l^{\circ} = du = \frac{x\,dz}{\sqrt{xx - zz}} = \frac{2\pi x}{nT} \times dt$, calling nT the time which the pole of the lunar orbit employs in performing its revolution in the circumference $2\pi x$, whence we get $dt = \frac{nT}{2\pi} \times \frac{dz}{\sqrt{xx - zz}}$.

The ſpace run through in an inſtant dt by the pole P of the earth parallel to the plane of the lunar orbit being by *Art.* 17. $= Bb \times \frac{aa}{2r} \times dt$; and by *Prob.* VIII. the ſpace run through parallel to the plane of the ecliptic, will be $Bb \times \frac{aa}{2rr} \times dt \times \frac{x\,I}{P\,x} =$ (by the preceding denominations) $Bb \times \frac{aa}{2rr} \times \frac{nT}{2\pi} \times \frac{dz}{\sqrt{xx - zz}}$

$$\times \frac{\frac{f}{a} \times \overline{g-z}}{\frac{ac}{r}} = B \times \frac{b}{r} \times \frac{nT}{4\pi} \times \frac{dz}{\sqrt{xx - zz}} \times \frac{f \times \overline{g-z}}{c};$$

and putting inſtead of $\frac{b}{r}$, its value, which was found to be $\frac{ff+gz}{ac}$, we ſhall have $B \times \frac{nT}{4\pi} \times \frac{f}{acc} \times \overline{ff+gz} \times \overline{g-z} \times \frac{dz}{\sqrt{xx - zz}}$; and when the point l° is on the other ſide with regard to l^{1}, we ſhall have $B \times \frac{nT}{4\pi} \times \frac{f}{acc} \times \overline{ff-gz} \times \overline{g+z} \times \frac{dz}{\sqrt{xx - zz}}$. Therefore always taking the preceſſion for the two points together placed on either ſide of l^{1}, and at an equal diſtance from it, we ſhall have, by joining the two values now found, after making reduction, $B \times \frac{nT}{4\pi} \times \frac{f}{aa} \times \overline{2gff - 2gzz} \times \frac{dz}{\sqrt{xx - zz}}$, or $B \times \frac{nT}{2\pi} \times \frac{fg}{aa} \times \overline{ff - zz} \times \frac{dz}{\sqrt{xx - zz}}$.

But the integral of $ff \times \frac{dz}{\sqrt{xx - zz}} = \frac{ff}{x} \times \frac{dz}{\sqrt{xx - zz}}$ is $\frac{ff}{x} \times \frac{x\pi}{2} = \frac{ff\pi}{2}$, when we take all the points l°, from l^{1} to l; and the integral of $\frac{zzdz}{\sqrt{xx - zz}}$, which is $- z\sqrt{xx - zz} + \int dz \sqrt{xx - zz}$; when, after the integration, we make $z = x$, becomes $\int dz \sqrt{xx - zz}$, or the area of the quadrant $l^{1} z l$, which is $= \frac{x\pi}{2} \times \frac{x}{2} = \frac{xx\pi}{4}$.

Therefore the integral of $B \times \frac{nT}{2\pi} \times \frac{fg}{acc} \times \overline{ff - zz} \times \frac{dz}{\sqrt{xx - zz}}$ is $= B \times \frac{nT}{2\pi} \times \frac{fg}{acc} \times \left(\frac{ff\pi}{2} - \frac{xx\pi}{4}\right) = B \times \frac{nT}{2} \times \frac{fg}{acc} \times \left(\frac{ff}{2} - \frac{xx}{4}\right) = B \times \frac{nT}{2} \times \frac{gf^{3}}{2acc} \times \left(1 - \frac{xx}{2ff}\right)$.

This is the ſum of all the inſtantaneous preceſſions for all places of the lunar orbit from l to l^{2}, becauſe we have always taken two points together placed on either ſide of l^{1}.

And this is the quantity of the preceſſion with regard to the ecliptic, during the time of a ſemi-revolution of the lunar orbit. Which was to be found.

PROBLEM X.

22. *To find the quantity of the nutation, or of the variation of the inclination of the earth's axe to the ecliptic, cauſed by the moon during the ſemi-revolution* $l\,l^1\,l^2$ *of the pole of the lunar orbit.*

The ſpace run through in an inſtant dt by the pole of the earth, parallel to the plane of the lunar orbit, being, by *Art.* 17. $= Bb \times \frac{aa}{2rr} \times dt$, the inſtantaneous nutation for any one place l° of the lunar orbit will be, by *Prob.* VIII. $= Bb \times \frac{aa}{2rr} \times dt \times \frac{PM}{PR}$ $=$ (by the denomination of the preceding *Problem*) $Bb \times \frac{aa}{2rr} \times \frac{nT}{2\pi} \times \frac{dz}{\sqrt{xx-zz}} \times \frac{\sqrt{xx-zz}}{\frac{ac}{r}} = B \times \frac{b}{r} \times \frac{aa}{2} \times \frac{nT}{2\pi} \times \frac{dz}{\sqrt{xx-zz}}$. Putting inſtead of $\frac{b}{r}$ its value, which was found to be $\frac{ff+gz}{ac}$, we ſhall have $B \times \frac{nT}{2\pi \times 2cc} \times \overline{ff+gz} \times dz$; and when the point l° is on the other ſide with regard to l^1, we ſhall have $B \times \frac{nT}{2\pi \times 2cc} \times \overline{ff-gz} \times dz$.

Taking therefore the nutation for the two points together, l°, placed on either ſide of l^1, we have

B ×

$B\times\frac{nT}{2\pi\times 2cc}\times 2ffdz$, whoſe integral, when z is $=x$, is $B\times\frac{nT}{2\pi\times 2cc}\times 2ffx = B\times\frac{nT}{2\pi}\times\frac{ff}{cc}\times x$.

This is the quantity of the nutation cauſed by the moon during the ſemi-revolution $l\,l^1\,l^2$ of the pole of the lunar orbit. Which was to be found.

COROLLARY.

23. By *Prob.* IX. the quantity of the preceſſion cauſed by the moon, with regard to the ecliptic, is, during the ſpace of a ſemi-revolution $l\,l^1\,l^2$ of the pole of the lunar orbit, $B\times\frac{nT}{2}\times\frac{gf^3}{2acc}\times\left(1-\frac{1}{2}\times\frac{xx}{ff}\right)$.

By the preceding *Problem*, the quantity of the nutation during the ſame time of a ſemi-revolution of the pole of the lunar orbit $l\,l^1\,l^2$, cauſed alſo by the moon, is $B\times\frac{nT}{2\pi}\times\frac{ff}{cc}\times x$.

Therefore the preceſſion, during the time of a ſemi-revolution of the pole of the lunar orbit, is to the nutation, during the ſame time,

As $\frac{gf}{2a}\times\left(1-\frac{1}{2}\,\frac{xx}{ff}\right)$, is to $\frac{x}{\pi}$; or as $gf\pi\times\left(1-\frac{1}{2}\,\frac{xx}{ff}\right)$ is to $2ax$.

But the preceſſion is performed in a parallel to the ecliptic, whereof g is the radius; and the nutation is performed in the plane of the ſolſticial colure, and in a circle whereof a is the radius: Dividing, therefore, each circular arc by its radius, we ſhall have the quantity of the angle: And

The angle of the preceſſion, during the time of a ſemi-revolution of $l\,l^1\,l^2$ of the pole of the lunar orbit, will be to the angle of the nutation, in the ſame time,

As $f\pi \times \left(1 - \frac{1}{2}\frac{xx}{ff}\right)$ is to $2x$, or as $\pi\left(1 - \frac{1}{2}\frac{xx}{ff}\right)$ is to $\frac{2x}{f}$.

SCHOLIUM.

24. Whatever has been ſaid as to the moon and the ſun, may be equally applicable to all the other planets; but as the effect of the action of the reſt of the planets upon the earth is not ſenſible, we ſhall not take it into conſideration.

SECT. III.

Of the motion of the pole of a ring, cauſed by the action of the ſun: or of the motion of the pole of the moon's orbit.

25. We have ſeen, in *Art.* 9. that if, inſtead of the terreſtrial ſpheroid, we were to ſuppoſe a ſimple ring placed at the circumference of the equator, the motion of the pole of ſuch ring would be found by the ſame *formula* as that for the motion of the pole of the earth; ſuppoſing only, in that *formula*, $\frac{a}{1-2a} = 1$.

By this means we may apply all the *formulæ* which have been found in *Sect.* I. to the motion of the nodes of this ring, and to the alteration of the inclination of its axe to the plane of the ecliptic.

But by the *Remark* on *Art.* 6. the motion of the plane of this ring is the ſame, whether the ring be entire, or there be only a ſingle point which circulates in the ring's circumference: Whence it follows, that

Theſe ſame *formulæ* do likewiſe give the motion of the pole of the orbit of a moon or ſatellite; the motion of the node of the orbit of ſuch moon in the ecliptic; and the variation of the inclination of its axe of the orbit to the plane of the ecliptic: Obſerving to put the time of the revolution of this moon about the ſun, inſtead of the time of the earth's revolution about the ſun, and the motion of the moon in its orbit inſtead of the motion of a point of the equator.

Remark.

It may be obſerved, that although the motion of the pole of a ring be the ſame as that of a moon, during the time of the revolution of the ring, or of the moon, which is the ſame; yet there are ſome particular motions which take place when there is only one moon revolving in the circumference of the ring, and which ceaſe to exiſt when it is an entire ring that revolves.

For example; in *Art.* 2. the force, according to $L\gamma$, *fig.* 1. n° 2. of the point L, is deſtroyed by an equal and directly oppoſite force of another point λ placed on the other ſide, and at the ſame diſtance from the point E, as the point L, when the ring is entire;

but

but when there is but one point, as *L*, or, which is the ſame, one moon, which revolves in the circumference of the ring; this motion ſubſiſts, and diſturbs the motion of the moon in the plane of its orbit.

In like manner the force of the point *L* parallel to ♈ *C*, or according to *c′ L*, is deſtroyed, in a ring, by the force of another equal point λ' placed on the other ſide with regard to ☊, and at the ſame diſtance as the point *L*. But this force ſubſiſts in the caſe where there is but one ſingle point *L*, or one moon, and this motion diſorders, beſides, the motion of the moon in the plane of its orbit.

Laſtly, If the ring, which circulates about the centre *C*, be elliptical, the motion of every point of the ring in the orbit is proportional to its diſtance from the centre; becauſe, on account of the continuity of the ring, all the points complete their revolution in the ſame time, and have equal angular motions; but when it is but one ſingle moon that revolves, in an elliptic orbit, it deſcribes areas proportional to the times, and has an angular motion ſo much the greater, as it approaches nearer the centre; which is the contrary of the motion of the points of the elliptic ring, the more diſtant whereof from the centre, have the moſt motion. And it ariſes from hence, that the motion of the pole of the lunar orbit is much the ſame as that of the ring, taking it during the entire revolution; but that there is a ſmall variation, or inequality, in the inſtantaneous motion of the pole of a moon; whereas this inſtantaneous motion is conſtant through every inſtant of the revolution of the ring.

I ſhall ſpend no more time on the ring, with reſpect to thoſe motions which have nothing to do with the ſubject in queſtion; but they ſhall be thoroughly diſcuſſed, and applied, in a ſecond memoir, which I ſhall publiſh, on the particular motions of the planets.

S E C T. IV.

Containing the application of the formulæ *of the preceding Sections.*

26. The motion of the ſun, or, to ſpeak more exactly, of the earth, in the ecliptic, is $\frac{2\pi s}{T}$, which is the ſpace divided by the time.

27. The earth's central force will therefore be $\left(\frac{2\pi s}{T}\right)^2 \times \frac{1}{2s}$, the central force being equal to the ſquare of the motion divided by the diameter of the circumference in which the body moves; and the earth's central force being the force $\frac{S}{ss}$ wherewith it is attracted towards the ſun; we ſhall have $\frac{S}{ss} = \left(\frac{2\pi s}{T}\right)^2 \times \frac{1}{2s} = \frac{2\pi\pi s}{TT}$; whence we get $\frac{S}{s^3} = \frac{2\pi\pi}{TT}$.

28. The motion of a point of the equator whoſe radius is a, will be, calling the time of its revolution t, $= \frac{2A\pi}{t}$, which is the ſpace divided by the time. We have therefore $m = \frac{2A\pi}{t}$.

29.

29. By *Art.* 8. $A = \frac{3S}{s^3 m} \times \frac{\mathrm{a}a}{1-2\mathrm{a}}$ = (by *Articles* 27. & 28.) $3 \times \frac{2\pi\pi}{TT \times 2\mathrm{a}\pi} \times \frac{\mathrm{a}a}{1-2\mathrm{a}} = \frac{3t}{TT} \times \pi \times \frac{\mathrm{a}}{1-2\mathrm{a}}$ = (by *Art.* 31. following) $\frac{3t}{TT} \times \pi \times \frac{1}{176}$.

30. Putting q for the ratio of the lunar force $\frac{L}{\lambda^3}$, to the ſolar force $\frac{S}{s^3}$, we ſhall have $\frac{L}{\lambda^3} = \frac{S}{s^3} \times q$; and B, which (by *Art.* 16.) is $= \frac{3L}{\lambda^3 m} \times \frac{\mathrm{a}a}{1-2\mathrm{a}}$ will alſo be $= \frac{3S}{s^3 m} \times q \times \frac{\mathrm{a}a}{1-2\mathrm{a}}$ = (by *Art.* 29.) $\frac{3qt}{TT} \times \pi \times \frac{1}{176}$.

31. Suppoſing $\mathrm{a} = \frac{1}{178}$, which is conformable to the obſervations of the gentlemen of the *Pariſian* academy, we ſhall have $\frac{\mathrm{a}}{1-2\mathrm{a}} = \frac{1}{178 \times \overline{1-\frac{2}{178}}} = \frac{1}{176}$.

PROBLEM XI.

32. *To find the preceſſion cauſed by the ſun during the time between the equinox and the ſolſtice; that is to ſay, in a quarter of a year, or three months.*

By *Art.* 12. the preceſſion, during this time, is $Ab \times \frac{T}{2} \times \frac{g}{4r}$. Putting, inſtead of A, the value found in *Art.* 29. we ſhall have $\frac{3t}{TT} \times \pi \times \frac{1}{176} \times \frac{bT}{2} \times \frac{g}{4r} = \frac{3t}{T} \times \frac{1}{176} \times \frac{b}{r} \times \frac{\pi g}{8}$.

But $\frac{b}{r}$ is the ratio of the ſine of the inclination of the axe of the earth to the ecliptic, to the radius, or of the ſine of 66 degrees 32 minutes to the radius, $= \frac{9172919}{10000600}$; and πg is the ſemi-circumference of the parallel to the ecliptic wherein the pole of the earth moves, and, conſequently, equal to 180 deg.

t, which is the time of the revolution of a point of the equator, is equal to one day; and T equal to $365\frac{1}{4}$ days; we ſhall have, then, by ſubſtituting all the values in the preceding *formulæ*, $\frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000} \times \frac{1}{176} \times \frac{180}{8}$ degrees. This is the preceſſion produced by the ſun during the ſpace of 3 months.

33. Multiplying by 4, we ſhall have the annual preceſſion cauſed by the ſun $= \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000} \times \frac{1}{176} \times \frac{180}{2}$ degrees, $= \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000} \times \frac{45}{88}$ degrees, = (after due reduction) $13''.\ 52'''.\ 11''''$.

PRO-

PROBLEM XII.

34. *To find the preceſſion cauſed by the moon, during the time of a ſemi-revolution of the pole of the lunar orbit about the pole of the ecliptic; that is, in about* $9\frac{1}{3}$ *years, or* $9\frac{1}{3} \times T$, *as* $T = 1$ *year.*

By *Art.* 21. we have found this preceſſion $= B \times \frac{nT}{2} \times \frac{gf^3}{2acc} \times \left(1 - \frac{1}{2} \times \frac{\varkappa\varkappa}{ff}\right)$.

By *Art.* 30. $B = \frac{3qt}{TT} \times \pi \times \frac{1}{176}$: Subſtituting this value in the preceding *formula*, it becomes $\frac{3qt}{T} \times \frac{n}{2} \times \frac{f}{2a} \times \frac{ff}{cc} \times \frac{1}{176} \pi g \times \left(1 - \frac{1}{2} \frac{\varkappa\varkappa}{ff}\right)$.

But $\frac{f}{a}$ is the ratio of the ſine of 66 deg. 32 min. to the radius, $= \frac{9172919}{10000000}$.

$\frac{f}{c}$ is the ratio of the ſine of 85 deg. to the radius, the angle lcz, whereby the pole of the lunar orbit is diſtant from the pole of the ecliptic, being, at all times, about 5 degrees. Therefore $\frac{f}{c} = \frac{9961947}{10000000}$.

$\frac{\varkappa}{f}$ is the ratio of the ſine of 5 deg. to the ſine of 85 degrees, $= \frac{871557}{9961947}$.

Therefore ſubſtituting theſe values, and thoſe of t, T, and πg found in the preceding *Problem*, we ſhall find that the preceſſion, during a ſemi-revolution of the lunar pole, which is $\frac{nT}{2}$, by *Art.* 21. becomes

$$= q \times \frac{n}{2} \times \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000 \times 2} \times \left(\frac{9961947}{10000000}\right)^2 \times \frac{1}{176}$$

$$\times\ 180 \text{ deg.} \times \left(1 - \tfrac{1}{2} \times \left(\frac{871557}{9961947}\right)^2\right).$$

This is the preceſſion cauſed by the moon during the time of a ſemi-revolution of the pole of the lunar orbit, $\frac{nT}{2} = 9\frac{1}{3} \times T$.

35. Therefore the mean preceſſion cauſed by the moon during the time T, which is the mean annual preceſſion cauſed by the moon, will be, by dividing the preceding *formula* by $\frac{n}{2}$, $= q \times \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000 \times 2}$

$$\times \left(\frac{9961947}{10000200}\right)^2 \times \frac{180 \text{deg.}}{176} \times \left(1 - \tfrac{1}{2} \times \left(\frac{871557}{9961947}\right)^2\right)$$

where it only remains to determine the value of q, which, by *Art.* 30. is the ratio of the lunar force to the ſolar force; and making this ratio $= \frac{5}{2}$, conformable to the obſervations of the tides related by Mr. *Daniel Bernoulli*, we ſhall have for the mean annual preceſſion cauſed by the moon, $\frac{5}{2} \times \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000 \times 2}$

$$\times \left(\frac{9961947}{10000000}\right)^2 \times \frac{180 \text{deg.}}{176} \times \left(1 - \tfrac{1}{2} \times \left(\frac{871557}{9961947}\right)^2\right)$$

= (after due reduction) $34''.\ 16'''.\ 51''''$.

PROBLEM XIII.

36. *To find the nutation of the earth's axe cauſed by the moon, during the time of a ſemi-revolution of the pole of the moon's orbit, that is, in $9\frac{1}{3}$ years.*

By *Art.* 23. $\pi \times \left(1 - \frac{1}{2} \times \frac{xx}{ff}\right)$ is to $\frac{2x}{f}$, as the angle of the preceſſion, ſound in the preceding *Problem* for the time $\frac{nT}{2}$ of the ſemi-revolution of the pole of the lunar orbit, is to the nutation during the ſame time. By it, therefore, may be found the nutation, having the quantity of the preceſſion of the preceding *Prob.* or, reciprocally, knowing the nutation during the time of a ſemi-revolution of the pole of the lunar orbit, the preceſſion will be known to a geometrical certainty, for the ſame time. The ratio, in numbers, is $\frac{1095798}{61236}$.

37. But we may alſo know the nutation during the time of a ſemi-revolution of the pole of the lunar orbit, by the *formula* of *Art.* 22. which is $B \times \frac{nT}{2} \times \frac{ff}{cc} \times x$; which, without alteration, may be multiplied by $\frac{a\pi}{a\pi}$, or by $\frac{a\pi}{a \times \frac{22}{7}}$; ſince π, which is the ratio of the circumference to the diameter, is $\frac{22}{7}$, and we ſhall

have the *formula* $B \times \frac{nT}{2} \times \frac{ff}{cc} \times x \times \frac{a\pi}{a \times \frac{22}{7}} = B \times \frac{nT}{2} \times \frac{f}{a} \times \frac{f}{c} \times \frac{x}{c} \times \frac{7}{22} \times a\pi$; which, by ſubſtituting the values of the preceding *Problem*, and 180 degrees inſtead of $a\pi$, which is the ſemi-circumference of the circle wherein the nutation is performed, becomes $\frac{3qt}{T} \times \frac{n}{2} \times \frac{9172919}{10000000} \times \frac{9961947}{10000000} \times \frac{871557}{10000000} \times \frac{7}{22} \times \frac{180}{176}$deg. $= \frac{3}{365\frac{1}{4}} \times \frac{5}{2} \times 9\frac{1}{3} \times \frac{9172919}{10000000} \times \frac{9961947}{10000000} \times \frac{871557}{10000000} \times \frac{7}{22} \times \frac{180}{176}$ deg. equal, after reduction, to $17''. 51'''. 14''''$.

PROBLEM XIV.

38. *To find the difference of the inclination of the earth's axe, between the point of the equinox and that of the ſolſtice, cauſed by the action of the ſun.*

By *Art.* 14. the *formula* is $\frac{1}{2} A \times \frac{T}{2\pi r} \times aa$.

Putting, inſtead of A, its value of *Art.* 29. inſtead of $\frac{a}{r}$, which is the ratio of the ſine of the obliquity of the ecliptic to the radius $\frac{3982155}{10000000}$, we ſhall have $\frac{1}{2} \times \frac{3}{365\frac{1}{4}} \times \frac{1}{2 \times 176} \times \frac{3982155}{10000000} \times a$; and putting, in-

inſtead of a, $\frac{a\pi}{\pi} = \frac{a\pi}{\frac{22}{7}} = \frac{7}{22}\,a\,\pi = \frac{7}{22} \times 180$ deg. (ſince $a\,\pi$ is the ſemi-circumference of the circle wherein this inclination is performed) we ſhall have $\frac{3}{365\frac{1}{4}} \times \frac{1}{4 \times 176} \times \frac{3982155}{10000000} \times \frac{7}{22} \times 180$ deg. $=$ (after reduction) $57'''.\ 25''''$.

PROBLEM XV.

39. *To find the difference of the inclination of the earth's axe to the plane of the lunar orbit, when the moon arrives at its point of ſtation, 90 degrees diſtant from the node of the equator with the lunar orbit, this difference of inclination being cauſed by the moon.*

By *Art.* 18. the variation of inclination will be $\frac{1}{2} B \times \frac{M}{2\pi r} \times a\,a$.

Putting, inſtead of B, its value of *Art.* 30.; in-inſtead of M, which expreſſes the time of the moon's revolution about the earth, $27\frac{1}{3}$ days; inſtead of $\frac{a}{r}$, the mean value $\frac{3982155}{10000000}$; I ſay *mean value*, becauſe the earth's axe is not always equally inclined to the plane of the lunar orbit, as has been ſhewn in *Art.* 19. *&c.* we ſhall have $\frac{1}{2} \times \frac{3 \times \frac{5}{2}}{(365\frac{1}{4})^2} \times \frac{27\frac{1}{3}}{2 \times 176} \times \frac{3982155}{10000000} \times a =$ (putting, as in the preceding *Problem*, inſtead of a, $\frac{7}{22} \times 180$ degrees) $\frac{1}{2} \times$ $\frac{3\frac{5}{7}}{(365\frac{1}{4})^2} \times \frac{27\frac{1}{3}}{2 \times 176} \times \frac{3982155}{10000000} \times \frac{7}{22} \times 180$ degrees, = (after reduction) $10'''. 19''''$.

PROBLEM XVI.

40. *To find the motion of the node of* Jupiter's *fourth ſatellite, cauſed by the ſun, during the time of* $\frac{1}{4}$ *of a revolution of that ſatellite about the ſun.*

We need only make uſe of the *formula* of *Problem* XI. wherein, conformably to *Art.* 25. we muſt make $\frac{a}{1-2a} = 1$, $t = 16\frac{3}{4}$ days, being the time of the revolution of the ſatellite about *Jupiter*; $T = 365\frac{1}{4} \times 12$ days, which is the time of the ſatellite's revolution about the ſun, or about 12 years; $\frac{b}{r}, = 1$, becauſe the orbit of the ſatellite is but little inclined to the ecliptic, and, conſequently, the axe thereof nearly at right angles thereto.

We ſhall have then $\frac{3t}{T} \times \frac{\pi g}{8}$, or $\frac{3 \times 16\frac{3}{4}}{365\frac{1}{4} \times 12} \times \frac{180 \text{ deg.}}{8}$.

This is the motion of the node of this ſatellite in 3 years, or during $\frac{1}{4}$ of its revolution about the ſun.

Remark.

41. By this may be found the motion of the nodes of the reſt of *Jupiter*'s ſatellites. For it appears, from the *formula*, that the motion of the nodes of

different ſatellites of the ſame planet, is as the times of their revolutions about the planet.

Thus the motion of the nodes of the 3d ſatellite, in the ſpace of 3 years, is $\frac{3 \times 7\frac{1}{6}}{365\frac{1}{4} \times 12} \times \frac{180 \text{ deg.}}{8}$.

The motion of the nodes of the 2d is $\frac{3 \times 3\frac{13}{24}}{365\frac{1}{4} \times 12} \times \frac{180 \text{ deg.}}{8}$.

The motion of the nodes of the 1ſt is $\frac{3 \times 1\frac{3}{4}}{365\frac{1}{4} \times 12} \times \frac{180 \text{ deg.}}{8}$.

PROBLEM XVII.

42. *To find the motion of the nodes of the 5th ſatellite of* Saturn, *cauſed by the ſun, during the time of* $\frac{1}{4}$ *of the revolution of this ſatellite about the ſun.*

Conformably to the preceding *Problem* we ſhall have, by putting inſtead of t, $79\frac{1}{3}$ days; inſtead of T $365\frac{1}{4} \times 30$ days, which is nearly the time of the revolution of this ſatellite about the ſun, and we ſhall have $\frac{3 \times 79\frac{1}{3}}{365\frac{1}{4} \times 30} \times \frac{180 \text{ deg.}}{8}$.

43. In the ſame manner may be found the motion of the nodes of all the reſt of *Saturn*'s ſatellites.

PROBLEM XVIII.

44. *To find the motion of the moon's node during the time that the ſun is paſſing from the node of the moon's orbit with the ecliptic, to its point of ſtation; that is, in three months, wanting a few days.*

Here we are to make uſe of the *formula* of *Art.* 13. becauſe the motion of the moon's nodes is very ſenſible in reſpect of the motion of the ſun.

We ſhall have $Ab \times \frac{T}{2r} \times \frac{g}{4} \times (1 - Ab \times \frac{T}{2\pi r} \times \frac{3}{4})$; and employing the values of *Prob.* XI. except that here $t = 27\frac{1}{3}$ days, which is the time of the moon's revolution about the earth, and that $\frac{b}{r} =$ to the ſine of 85 deg. divided by the radius; the axe of the lunar orbit being inclined in an angle of about 85 degrees to the ecliptic. Therefore $\frac{b}{r} = \frac{9961947}{10000000}$. Laſtly, by *Art.* 25. we muſt take $\frac{a}{1 - 2a} = 1$, and the preceding *formula* will be changed into this, $\frac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \frac{9661947}{10000000} \times \frac{180 \text{ deg.}}{8} \times (1 - \frac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \frac{9961947}{10000000} \times \frac{3}{2 \times 8}) =$ (after reduction) $4^{\circ}.\ 36'.\ 33''$.

This is the motion of the moon's node during the time that the ſun is paſſing from the node of the moon's orbit with the ecliptic to its point of ſtation; that is, during the time the ſun is moving 90 — 4 degrees, 36 minutes, 33 ſeconds, or 85°. 23′. 27″. And if the motion of the node be required for one year, 'tis only multiplying the motion, juſt found, by $\frac{360^\circ}{85^\circ.\,23'.\,27''}$, and it will be found 19°. 25′. 39″. being a very ſmall matter more than the truth, for reaſons which will be explained elſewhere, and becauſe we have taken 27⅓ days for the time of the moon's revolution about the earth, which is really a little leſs than that quantity.

Problem XIX.

45. *To find the greateſt difference of the inclination of the lunar orbit to the ecliptic; which happens when the ſun arrives at his point of ſtation with regard to the moon.*

Here the *formula* of *Art.* 15. muſt be made uſe of, which gives $\frac{1}{2}Aaa \times \frac{T}{2\pi r} \times \left(1 - \frac{1}{2}Ab \times \frac{T}{2\pi r}\right)$, or, which is the ſame, $\frac{1}{2}A \times \frac{aT}{2\pi r} \times \frac{a\pi}{\frac{22}{7}} \times \left(1 - \frac{1}{2}A \times \frac{bT}{2\pi r}\right)$, ſubſtituting the values as in the preceding *Problem*, and inſtead of $\frac{a}{r}$, putting $\frac{871557}{10000000}$, which is the ratio of the ſine of 5° to radius, we ſhall have $\frac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \frac{1}{2 \times 2} \times \frac{871557}{10000000} \times \frac{7}{22} \times 180 \text{ deg.} \times \left(1 - \frac{1}{4} \times \frac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \frac{9961947}{10000000}\right) = 15'.\,51''.$

Remark.

46. It ſhould be obſerved, that, to ſhorten the computations, I have contented myſelf with taking the times of the revolutions pretty near the truth; but if the utmoſt exactneſs be required, the accurate times of the revolutions muſt be employed.

47. This might be a proper place to add the method of determining the perturbation of the orbit of any planet, as derived from another planet; but ſince this depends upon no other than the very ſame principles that have been made uſe of in this memoir, and as their application will be ſhewn, in its full extent, in the memoir which I am going to print, and intend myſelf the honour of ſending to the *Royal Society*, I ſhall deſiſt, that I may not run this paper to a greater length.

LIX. *A Letter to the Right Honourable* George *Earl of* Macclesfield, *P. R. S. concerning the ages of* Homer *and* Heſiod. *By* George Coſtard, *M. A.*

My Lord,

Read Dec. 13, 1753.

IT ſeems to be an opinion pretty generally received, that Homer and Heſiod lived much about the ſame time. If this be true, and they did ſo, whatever arguments prove the age of one, will equally ſerve for fixing that of the other. What that age was, is indeed not at all agreed on among writers; the only thing in which they conſpire

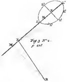
Fig 3 N° 2
p 416

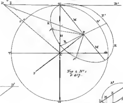
Fig 2 N° 1
p 407

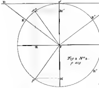
Fig 2 N° 2
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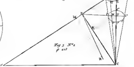
Fig 3 N° 1
p 413

J Mynde Sculp